9 and 10 GRADE. PROBLEM SET 1

1. A passenger on a bus looks out at the window and sees his friend walking in the opposite direction. He gets off at the next stop, 3 minutes after having seen his friend, and starts walking in the opposite direction to catch up with him. How much time does he need for it (counting from the moment he got off the bus) if he walks 2.5 times faster than his friend but 6 times slower than the bus. (All speeds are constant.)

   Answer: 32 minutes.

   Solution. Let v be the speed of a friend. Then speed of a former bus passenger is equal to $2.5v$, and the speed of the bus is equal to $6 \cdot 2.5v = 15v$. In 3 minutes that pass after the passenger has seen his friend the distance between them becomes equal to $3 \cdot (15v + v) = 48v$. When the passenger starts overtaking his friend, he reduces the distance between them by $1.5v$ every minute. So he needs $\frac{48v}{1.5v} = 32$ minutes for it.

2. Jack and Jill exchanged some stamps from their collections two times. Each time Jill gave $\frac{2}{11}$ of all her stamps to Jack and Jack gave one half of his stamps to Jill (for example, if Jill had 11 stamps and Jack had 10 stamps, it means that in the first exchange Jill would give 2 stamps to Jack and Jack would give 5 stamps to Jill). It turned out that after the first exchange Jack had 110 stamps and after the second exchange Jill had 334 stamps. How many stamps did Jill have before all the exchanges?

   Answer: 363. // And Jack had 88.

   Solution. Let the initial quantities of stamps in Jack's and Jill's possession be equal to x and y respectively. Then after the first exchange Jack has $\frac{1}{2}x + \frac{2}{11}y$ stamps and Jill has $\frac{1}{2}x + \frac{9}{11}y$ stamps. After the second exchange these numbers are equal to $\frac{15}{44}x + \frac{29}{121}y$ and $\frac{29}{44}x + \frac{92}{121}y$ respectively. It is known that $\frac{1}{2}x + \frac{2}{11}y = 110$ and $\frac{29}{44}x + \frac{92}{121}y = 334$. Therefore, $x = 88$ and $y = 363$.

3. An irreducible fraction, its numerator and denominator being positive integers, is greater than $\frac{1}{11}$. If its denominator is increased by 1 and its numerator is increased by 6 the resulting fraction is less than 0.2. Find the initial fraction if it is known that its denominator is 8 less than the square of its numerator.

   Answer: $\frac{11}{113}$.

   Solution. Let the numerator of the fraction be equal to k. Then its denominator is $k^2 - 8$. It is known that $\frac{k}{k^2-8} > \frac{1}{11}$ and $\frac{k+6}{k^2-7} < \frac{1}{5}$. As the denominator of a fraction is positive, and its numerator is integer, we deduce that $k \geq 3$. Therefore, we can multiply both parts of the first inequality by a *positive* expression $11\left(k^2-8\right)$ and multiply both parts of the second by a *positive* expression $5\left(k^2-7\right)$, thus getting a system

$$\begin{cases} k^2 - 11k - 8 < 0, \\ k^2 - 5k - 37 > 0. \end{cases}$$

   Taking into account that k is an integer greater than 1, the first inequality yields $3 \leq k \leq 11$, and from the second inequality it follows that $k \geq 10$. Combining the two conditions we get that either $k = 10$ or $k = 11$. If $k = 10$ the initial fraction is equal to $\frac{10}{92}$, so it is reducible by 2. If $k = 11$ then the fraction equals $\frac{11}{113}$ that is irreducible. It is the only possibility that satisfies all the given conditions.

4. Solve the system of equations $\begin{cases} xy^3 - x = 182, \\ xy^2 - xy = 42. \end{cases}$

   Answer: $(7; 3)$, $(-189; \frac{1}{3})$.

Solution. We start with factorizing the left sides of both equations:

$$\begin{cases} x(y-1)\left(1+y+y^2\right) = 182, \\ xy(y-1) = 42. \end{cases}$$

As the right sides of the equations are different from zeroes (and left sides are equal to them) we can divide the first equation by the second one; we get $\frac{1+y+y^2}{y} = \frac{13}{3}$, $3y^2 - 10y + 3 = 0$, therefore, $y = 3$ or $y = \frac{1}{3}$. Substituting these values into the first equation of the initial system we can find the respective values of x:

if $y = 3$ then $9x - 3x = 42 \Leftrightarrow x = 7$;

if $y = \frac{1}{3}$ then $\frac{x}{9} - \frac{x}{3} = 42 \Leftrightarrow x = -189$.

5. The perimeter of a right triangle ABC is equal to 30. Hypotenuse AB touches the incircle of this triangle at point Q, and $AQ : QB = 10 : 3$. Find the area of this triangle.

   Answer: 30.

   Answer. Let us denote touch points of the incircle with sides AC and BC as D and F respectively. Let $AQ = 10x$, $CD = y$. As $AQ : QB = 10 : 3$, we get that $BQ = 3x$. We also get $AD = AQ = 10x$, $CF = CD = y$, $BF = BQ = 3x$ since the segments of tangent lines drawn to a circle from one point are equal to each other. As perimeter of the triangle is 30, we get that $2y + 26x = 30$, $y = 15 - 13x$. So, $AC = 10x + y = 15 - 3x$, $BC = 3x + y = 15 - 10x$, $AB = 3x + 10x = 13x$. Then Pythagorean theorem yields $(15-3x)^2 + (15-10x)^2 = (13x)^2 \Leftrightarrow 2x^2 + 13x - 15 = 0$, and so $x = 1$ or $x = -\frac{15}{2}$. The negative value of x is not suitable. Thus, $x = 1$, $AC = 12$, $BC = 5$, $S_{\triangle ABC} = \frac{1}{2} AC \cdot BC = 30$.

6. Find all pairs of integers $(x; y)$ that satisfy inequalities $x^2 + 16y + 193 < 24x - y^2$ and $38x - y^2 > x^2 + 8y + 354$.

   Answer: $(15; -6)$.

   Solution. Making out the exact squares we get $\begin{cases} (x-12)^2 + (y+8)^2 < 15, \\ (x-19)^2 + (y+4)^2 < 23 \end{cases}$. As squares are non-negative it follows from these inequalities that $(x-12)^2 < 15$ and $(x-19)^2 < 23$. The only integer value of x that satisfies these inequalities is $x = 15$. Substituting it into the system we get $\begin{cases} (y+8)^2 < 6, \\ (y+4)^2 < 7 \end{cases}$. The only value of y that suits here is $y = -6$. So we got the only pair of integers $(15; -6)$ that satisfy the given inequalities.

7. KT is a bisector of triangle KLM. Circle Ω with its center on side KM has a radius of 6 and passes through points K, L and T. Find KL if it is known that $LT : MT = 1 : 3$.

   Answer: 8.

   Solution. Let $LT = y$. Then $MT = 3y$. As KT is a bisector of a triangle, $KL : KM = LT : MT = y : 3y = 1 : 3$. Let $KL = x$, then $KM = 3x$. ML and MK are two secant lines to a circle drawn from one point; therefore, $MT \cdot ML = MK \cdot MP$ (P is the intersection point of KM with the circle), i.e. $3y \cdot 4y = 3x \cdot (3x - 12) \Leftrightarrow 4y^2 = 3x^2 - 12x$. As KP is a diameter, it subtends a right angle at point L, and so $\cos \angle LKP = \frac{KL}{KP} = \frac{x}{12}$. Then cosine theorem for triangle KLM yields $(4y)^2 = (x)^2 + (3x)^2 - 2 \cdot x \cdot 3x \cdot \frac{x}{12}$. Substituting here the expression for $4y^2$ obtained above we get $12x^2 - 48x = 10x^2 - \frac{1}{2}x^3 \Leftrightarrow x(x-8)(x+12) = 0$. This equation has the only positive root, that is $x = 8$. Consequently, $KL = x = 8$.

9 and 10 GRADE. PROBLEM SET 2

1. A passenger on a bus looks out at the window and sees his friend walking in the opposite direction. He gets off at the next stop, 1.5 minutes after having seen his friend, and starts walking in the opposite direction to catch up with him. How much time does he need for it (counting from the moment he got off the bus) if he walks 1.8 times faster than his friend but 11 times slower than the bus. (All speeds are constant.)

   Answer: 39 minutes.

   Solution. Let v be the speed of a friend. Then speed of a former bus passenger is equal to $1.8v$, and the speed of the bus is equal to $11 \cdot 1.8v = 19.8v$. In 1.5 minutes that pass after the passenger has seen his friend the distance between them becomes equal to $1.5 \cdot (19.8v + v) = 31.2v$. When the passenger starts overtaking his friend, he reduces the distance between them by $0.8v$ every minute. So he needs $\frac{31.2v}{0.8v} = 39$ minutes for it.

2. Jack and Jill exchanged some stamps from their collections two times. Each time Jill gave $\frac{3}{7}$ of all her stamps to Jack and Jack gave one third of his stamps to Jill (for example, if Jill had 14 stamps and Jack had 12 stamps, it means that in the first exchange Jill would give 6 stamps to Jack and Jack would give 4 stamps to Jill). It turned out that after the first exchange Jack had 273 stamps and after the second exchange Jill had 199 stamps. How many stamps did Jill have before all the exchanges?

   Answer: 147.

   Solution. Let the initial quantities of stamps in Jack's and Jill's possession be equal to x and y respectively. Then after the first exchange Jack has $\frac{2}{3}x + \frac{3}{7}y$ stamps and Jill has $\frac{1}{3}x + \frac{4}{7}y$ stamps. After the second exchange these numbers are equal to $\frac{37}{63}x + \frac{26}{49}y$ and $\frac{26}{63}x + \frac{23}{49}y$ respectively. It is known that $\frac{2}{3}x + \frac{3}{7}y = 273$ and $\frac{26}{63}x + \frac{23}{49}y = 199$. Therefore, $x = 315$ and $y = 147$.

3. An irreducible fraction, its numerator and denominator being positive integers, is greater than $\frac{1}{9}$. If its numerator is increased by 3 and its denominator is increased by 1 the resulting fraction is less than 0.2. Find the initial fraction if it is known that its denominator is 3 less than the square of its numerator.

   Answer: $\frac{8}{61}$.

   Solution. Let the numerator of the fraction be equal to k. Then its denominator is k^2-3. It is known that $\frac{k}{k^2-3} > \frac{1}{9}$ and $\frac{k+3}{k^2-2} < \frac{1}{5}$. As the denominator of a fraction is positive, and its numerator is integer, we deduce that $k \geq 2$. Therefore, we can multiply both parts of the first inequality by a *positive* expression $9\left(k^2-3\right)$ and multiply both parts of the second by a *positive* expression $5\left(k^2-2\right)$, thus getting a system

$$\begin{cases} k^2 - 9k - 3 < 0, \\ k^2 - 5k - 17 > 0. \end{cases}$$

   Taking into account that k is an integer greater than 1, the first inequality yields $2 \leq k \leq 9$, and from the second inequality it follows that $k \geq 8$. Combining the two conditions we get that either $k = 8$ or $k = 9$. If $k = 9$ the initial fraction is equal to $\frac{9}{78}$, so it is reducible by 3. If $k = 8$ then the fraction equals $\frac{8}{61}$ that is irreducible. It is the only possibility that satisfies all the given conditions.

4. Solve the system of equations $\begin{cases} x + xy^3 = -70, \\ xy + xy^2 = 20. \end{cases}$

   Answer: $(10; -2)$, $(-80; -\frac{1}{2})$.

Solution. We start with factorizing the left sides of both equations:

$$\begin{cases} x(1+y)\left(1-y+y^2\right) = -70, \\ xy(1+y) = 20. \end{cases}$$

As the right sides of the equations are different from zeroes (and left sides are equal to them) we can divide the first equation by the second one; we get $\frac{1-y+y^2}{y} = -\frac{7}{2}$, $2y^2+5y+2=0$, therefore, $y=-2$ or $y=-\frac{1}{2}$. Substituting these values into the first equation of the initial system we can find the respective values of x:

if $y=-2$ then $x-8x=-70 \Leftrightarrow x=10$;

if $y=-\frac{1}{2}$ then $x-\frac{1}{8}x=-70 \Leftrightarrow x=80$.

5. The perimeter of a right triangle ABC is equal to 40. Hypotenuse AB touches the incircle of this triangle at point Q, and $AQ:QB=5:12$. Find the area of this triangle.

   Answer: 60.

   Answer. Let us denote touch points of the incircle with sides AC and BC as D and F respectively. Let $AQ=5x$, $CD=y$. As $AQ:QB=5:12$, we get that $BQ=12x$. We also get $AD=AQ=5x$, $CF=CD=y$, $BF=BQ=12x$ since the segments of tangent lines drawn to a circle from one point are equal to each other. As perimeter of the triangle is 40, we get that $2y+34x=40$, $y=20-17x$. So, $AC=5x+y=20-12x$, $BC=12x+y=20-5x$, $AB=5x+12x=17x$. Then Pythagorean theorem yields $(20-5x)^2+(20-12x)^2=(17x)^2 \Leftrightarrow 3x^2+17x-20=0$, and so $x=1$ or $x=-\frac{20}{3}$. The negative value of x is not suitable. Thus, $x=1$, $AC=8$, $BC=15$, $S_{\triangle ABC}=\frac{1}{2}AC\cdot BC=60$.

6. Find all pairs of integers $(x;\,y)$ that satisfy inequalities $x^2+26y+159<4x-y^2$ and $18x-y^2>x^2+18y+140$.

   Answer: $(5;-11)$.

   Solution. Making out the exact squares we get $\begin{cases}(x-2)^2+(y+13)^2<14, \\ (x-9)^2+(y+9)^2<22\end{cases}$. As squares are non-negative it follows from these inequalities that $(x-2)^2<14$ and $(x-9)^2<22$. The only integer value of x that satisfies these inequalities is $x=5$. Substituting it into the system we get $\begin{cases}(y+13)^2<5, \\ (y+9)^2<6\end{cases}$. The only value of y that suits here is $y=-11$. So we got the only pair of integers $(5;-11)$ that satisfy the given inequalities.

7. KT is a bisector of triangle KLM. Circle Ω with its center on side KM has a radius of 6 and passes through points K, L and T. Find KL if it is known that $LT:MT=2:3$.

   Answer: 10.

   Solution. Let $LT=2y$. Then $MT=3y$. As KT is a bisector of a triangle, $KL:KM=LT:MT=2y:3y=2:3$. Let $KL=2x$, then $KM=3x$. ML and MK are two secant lines to a circle drawn from one point; therefore, $MT\cdot ML=MK\cdot MP$ (P is the intersection point of KM with the circle), i.e. $3y\cdot 5y=3x\cdot(3x-12) \Leftrightarrow 5y^2=3x^2-12x$. As KP is a diameter, it subtends a right angle at point L, and so $\cos\angle LKP=\frac{KL}{KP}=\frac{x}{6}$. Then cosine theorem for triangle KLM yields $(5y)^2=(2x)^2+(3x)^2-2\cdot 2x\cdot 3x\cdot\frac{x}{6}$. Substituting here the expression for $5y^2$ obtained above we get $15x^2-60x=13x^2-2x^3 \Leftrightarrow x(x-5)(x+6)=0$. This equation has the only positive root, that is $x=5$. Consequently, $KL=2x=10$.

11 GRADE. PROBLEM SET 3

1. Solve the equation $x^4 \cdot 2^{11-x} + 2^{2+\sqrt{2x+2}} = x^4 \cdot 2^{\sqrt{2x+2}} + 2^{13-x}$.

   Answer: 7; $\sqrt{2}$.

   Solution. Moving all terms to the left side and factorizing, we get $(x^4-4)\left(2^{11-x}-2^{\sqrt{2x+2}}\right)=0$. The first factor is equal to 0 for $x=\pm\sqrt{2}$, and only $x=\sqrt{2}$ belongs to the domain of the second factor. The second factor equals 0 if

$$\sqrt{2+2x} = 11-x \Leftrightarrow \begin{cases} 2+2x = x^2-22x+121, \\ 11-x \geq 0 \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} x=17, \\ x=7, \end{array}\right. \\ x \leq 11 \end{cases} \Leftrightarrow x=7.$$

   Finally we get $x=7$ and $x=\sqrt{2}$.

2. Numbers a, b and c (in the indicated order) form a geometric progression. Numbers $c-a$, $2a-b$ and $a+b$ (in the indicated order) form an arithmetic progression. Find the common ratio of the geometric progression.

   Answer: -4 or 1.

   Solution. As $c-a$, $2a-b$ and $a+b$ form an arithmetic progression, its middle term is equal to half-sum of the other two terms; therefore, $2(2a-b)=(c-a)+(a+b) \Leftrightarrow c=4a-3b$. For the sequence to be a geometric progression, any of its terms squared should be equal to the product of the two neighbors. So, $b^2=ac$, or, taking into account that $c=4a-3b$, we get $a(4a-3b)=b^2$, $b^2+3ab-4a^2=0$. Solving this equation as quadratic equation with respect to b, we obtain that either $b=-4a$ or $b=a$. The common ratio q of a geometric progression is equal to $\frac{b}{a}$, and so we get that either $q=-4$ or $q=1$.

3. Find the value of $\cos 10^\circ \cdot \cos 50^\circ \cdot \cos 70^\circ$.

   Answer: $\frac{\sqrt{3}}{8}$.

   Solution. Transforming product of trigonometric functions into their sum we get $\cos 10^\circ \cos 50^\circ = \frac{1}{2}\cos 40^\circ + \frac{1}{2}\cos 60^\circ = \frac{1}{2}\cos 40^\circ + \frac{1}{4}$. Therefore, the initial expression is equal to $\left(\frac{1}{2}\cos 40^\circ + \frac{1}{4}\right) cos 70^\circ = \frac{1}{2}\cos 40^\circ \cos 70^\circ + \frac{1}{4}\cos 70^\circ = \frac{1}{4}\cos 110^\circ + \frac{1}{4}\cos 30^\circ + \frac{1}{4}\cos 70^\circ$. As $\cos 110^\circ = \cos\left(180^\circ - 70^\circ\right) = -\cos 70^\circ$, we finally get $\frac{1}{4}\cos 30^\circ = \frac{\sqrt{3}}{8}$.

4. Find the coordinates of point M such that it lies on y-axis and tangent lines drawn from point M to parabola $y=7-5x-3x^2$ are perpendicular to each other.

   Answer: $\left(0; \frac{55}{6}\right)$.

   Solution. Let us choose a point with ordinate a on y-axis. It is obvious that the lines we are considering are not parallel to coordinate axes. Then product of slopes of these lines equals (-1). Let us denote these slopes as k and $-\frac{1}{k}$ respectively. Consequently, the equations of the lines are $y=kx+a$ and $y=-\frac{1}{k}x+a$. Each of them has exactly one common point with the parabola, therefore systems

$$\begin{cases} y=7-5x-3x^2, \\ y=kx+a \end{cases} \quad \text{and} \quad \begin{cases} y=7-5x-3x^2, \\ y=-\frac{1}{k}x+a \end{cases}$$

   have one solution each. Equating the right sides of the equations in each of the systems, we get that equations

$$kx+a=7-5x-3x^2 \quad \text{and} \quad -\frac{1}{k}x+a=7-5x-3x^2$$

   must have exactly one solution each.

Let us consider the first equation separately. It is equivalent to $3x^2+(k+5)x+(a-7)=0$. For it to have one solution, its discriminant has to be 0, and so $(k+5)^2-12(a-7)=0$. In the same way second equation yields $\left(\frac{1}{k}-5\right)^2-12(a-7)=0$.

As k and a must satisfy both of the obtained equations, we can subtract the second of them from the first, thus getting $(k+5)^2-\left(\frac{1}{k}-5\right)^2=0$; hence $k+5=-\frac{1}{k}+5$ or $k+5=\frac{1}{k}-5$. The former equation yields $k^2=-1$, and so it has no solutions. The latter equation is equivalent to $k^2+10k-1=0$, thus $k=-5\pm\sqrt{26}$. Substituting these values of k into $(k+5)^2=12(a-7)$ we get $12(a-7)=26$, $a=\frac{55}{6}$.

5. Point P belongs to side LM of parallelogram $KLMN$. It is known that $LP=MP=2$, $\angle KPN=\arccos\frac{11}{12}$, $KL=9$. Find the area of this parallelogram.

Answer: $7\sqrt{23}$.

Solution. Let $\angle KLM=\varphi$, then $\angle LMN=180^\circ-\varphi$. Cosine theorem applied to triangles KLP and MNP yields $KP^2=81+4-36\cos\varphi$, $NP^2=81+4+36\cos\varphi$. Then we use cosine theorem for triangle KPN:

$$KN^2=(85+36\cos\varphi)+(85-36\cos\varphi)-2\sqrt{(85+36\cos\varphi)(85-36\cos\varphi)}\cdot\frac{11}{12}$$
$$\Leftrightarrow\sqrt{85^2-36^2\cos^2\varphi}=84\Leftrightarrow\cos^2\varphi=\frac{13^2}{36^2}.$$

Hence $\sin^2\varphi=\left(1-\frac{13}{36}\right)\left(1+\frac{13}{36}\right)=\frac{49\cdot 23}{36^2}$, $\sin\varphi=\frac{7\sqrt{23}}{36}$, and so the area of the parallelogram is equal to $KL\cdot LM\cdot\sin\varphi=7\sqrt{23}$.

6. Sketch the set of points whose coordinates $(x;\,y)$ satisfy the system of inequalities

$$\begin{cases}\log_{|y-2|-2|y-4|+6}(x+3)>\log_{|y-2|-2|y-4|+6}(1+y),\\ x<13.\end{cases}$$

Find the area of this set.

Answer: The set is a union of three trapezoids $ABCD$, $CKLM$ and $MPQR$, their borders do not belong to the set; the coordinates of the vertices are $A(-3;\,0)$, $B(-3;\,1)$, $C(-1;\,1)$, $D(-2;\,0)$, $K(13;\,1)$, $L(13;\,11)$, $M(9;\,11)$, $P(10;\,12)$, $Q(-3;\,12)$, $R(-3;\,11)$. The area of this set is 104.

Solution. Let us consider the two options available.

(a) The base of the logarithm is greater than 1. It happens if and only if

$$|y-2|-2|y+4|+6>1\Leftrightarrow|2y-8|<|y-2|+5\Leftrightarrow\begin{cases}2y-8<|y-2|+5,\\ 2y-8>-|y-2|-5\end{cases}\Leftrightarrow$$

$$\begin{cases}|y-2|>2y-13,\\ |y-2|>3-2y\end{cases}\Leftrightarrow\begin{cases}\left[\begin{array}{l}y-2>2y-13,\\ y-2<13-2y,\end{array}\right.\\ \left[\begin{array}{l}y-2>3-2y,\\ y-2<2y-3\end{array}\right.\end{cases}\Leftrightarrow\begin{cases}\left[\begin{array}{l}y<11,\\ y<5,\end{array}\right.\\ \left[\begin{array}{l}y>\frac{5}{3},\\ y>1\end{array}\right.\end{cases}\Leftrightarrow\begin{cases}y<11,\\ y>1\end{cases}\Leftrightarrow 1<y<11.$$

We get that $x+3>1+y$, i.e. $y<x+2$. If we also take into account that $x<13$, we obtain a trapezoid $CKLM$, the coordinates of its vertices being $C(-1;\,1)$, $K(13;\,1)$, $L(13;\,11)$, $M(9;\,11)$.

(b) The base of the logarithm is between 0 and 1. To determine the values of y for which it happens we solve the inequality $|y-2|-2|y+4|+6>0$ and then exclude an interval $[1;\,11]$ from its solution set.

$$|y-2|-2|y+4|+6>0 \Leftrightarrow |2y-8|<|y-2|+6 \Leftrightarrow \begin{cases} 2y-8<|y-2|+6, \\ 2y-8>-|y-2|-6 \end{cases} \Leftrightarrow$$

$$\begin{cases} |y-2|>2y-14, \\ |y-2|>2-2y \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} y-2>2y-14, \\ y-2<14-2y, \end{array}\right. \\ \left[\begin{array}{l} y-2>2-2y, \\ y-2<2y-2 \end{array}\right. \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} y<12, \\ y<\dfrac{16}{3}, \end{array}\right. \\ \left[\begin{array}{l} y>\dfrac{4}{3}, \\ y>0 \end{array}\right. \end{cases} \Leftrightarrow \begin{cases} y<12, \\ y>0 \end{cases} \Leftrightarrow 0<y<12.$$

Thus the base of the logarithm is between 0 and 1 for $y \in (0;\,1) \cup (11;\,12)$. For these values of y we conclude that $x+3<1+y$, i.e. $y>x+2$. If we add that $x>-3$ (that follows from the domain of the first inequality), we obtain two trapezoids $ABCD$ and $MPQR$, their vertices situated in points $A(-3;\,0)$, $B(-3;\,1)$, $C(-1;\,1)$, $D(-2;\,0)$, $M(9;\,11)$, $P(10;\,12)$, $Q(-3;\,12)$, $R(-3;\,11)$.

The area of this set is equal to sum of areas of all the trapezoids. $S_{ABCD} = \frac{1+2}{2} \cdot 1 = 1{,}5$, $S_{CKLM} = \frac{14+4}{2} \cdot 10 = 90$, $S_{ABCD} = \frac{12+13}{2} \cdot 1 = 12{,}5$; hence $S_{\text{total}} = 104$.

7. Isosceles trapezoids $APRS$ and $PQRS$, their largest bases being PR and PS respectively ($PR = PS$), are inscribed into circle Ω. Diagonals of trapezoid $PQRS$ intersect at point O, and angle POS equals 120°. Find the radius of Ω given that the area of triangle APS is equal to $4+4\sqrt{3}$.

Answer: 4.

Solution. Angle QOP is 60° and is the angle between the chords that intersect; therefore, it is equal to one half of the sum of arcs PQ and RS. As these arcs are equal to each other (they are arcs between parallel chords QR and PS), we conclude that each of them is equal to 60°. Chords AS and PR are also parallel to each other (as they are bases of a trapezoid), and so arc AP is also equal to 60°. Chords PR and PS are equal to each other; and so are the corresponding arcs, that is, arc PQR equals arc PAS. From here follows that $\smile QR = \smile AS$. As the whole circle is 360°, we get that $\smile QR = \smile AS = 90^\circ$.

Angle APS is inscribed into circle Ω, and so it is equal to $\frac{1}{2} \smile AS = \frac{1}{2} 90^\circ = 45^\circ$. In the same way $\angle APS = \frac{1}{2} \smile AP = 30^\circ$. Then $\angle PAS = 180^\circ - \angle APS - \angle ASP = 105^\circ$. Let the diameter of the circle be equal to d. Then sine theorem used for triangle APS yields $AP = d \sin 30^\circ = \frac{d}{2}$, $AS = d \sin 45^\circ = \frac{d}{\sqrt{2}}$. For expressing the area of triangle APS we also need that $\sin 105^\circ = \sin(45^\circ + 60^\circ) = \sin 45^\circ \cos 60^\circ + \cos 45^\circ \sin 60^\circ = \frac{1+\sqrt{3}}{2\sqrt{2}}$.

Hence area of triangle APS is equal to $\frac{1}{2} AP \cdot AS \cdot \sin 105^\circ = \frac{d^2\left(\sqrt{3}+1\right)}{16}$. Equating it to $4\left(\sqrt{3}+1\right)$ yields $d^2 = 64$, so $d = 8$ and the radius is equal to 4.

11 GRADE. PROBLEM SET 4

1. Solve the equation $x^4 \cdot 3^{\sqrt{1-3x}} + 3^{x+11} = x^4 \cdot 3^{x+9} + 3^{2+\sqrt{1-3x}}$.

   Answer: -5; $-\sqrt{3}$.

   Solution. Moving all terms to the left side and factorizing, we get $(x^4-9)\left(3^{\sqrt{1-3x}} - 3^{x+9}\right) = 0$. The first factor is equal to 0 for $x = \pm\sqrt{3}$, and only $x = -\sqrt{3}$ belongs to the domain of the second factor. The second factor equals 0 if

$$\sqrt{1-3x} = x+9 \Leftrightarrow \begin{cases} 1-3x = x^2+18x+81, \\ x+9 \geq 0 \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} x=-16, \\ x=-5, \end{array}\right. \\ x \geq -9 \end{cases} \Leftrightarrow x=-5.$$

   Finally we get $x=-5$ and $x=-\sqrt{3}$.

2. Numbers a, b and c (in the indicated order) form a geometric progression. Numbers $3c-2a$, $a-b$ and $a-2c$ (in the indicated order) form an arithmetic progression. Find the common ratio of the geometric progression.

   Answer: -3 or 1.

   Solution. As $3c-2a$, $a-b$ and $a-2c$ form an arithmetic progression, its middle term is equal to half-sum of the other two terms; therefore, $2(a-b) = (a-2c)+(3c-2a) \Leftrightarrow c = 3a-2b$. For the sequence to be a geometric progression, any of its terms squared should be equal to the product of the two neighbors. So, $b^2 = ac$, or, taking into account that $c = 3a-2b$, we get $a(3a-2b) = b^2$, $b^2+2ab-3a^2 = 0$. Solving this equation as quadratic equation with respect to b, we obtain that either $b=-3a$ or $b=a$. The common ratio q of a geometric progression is equal to $\frac{b}{a}$, and so we get that either $q=-3$ or $q=1$.

3. Find the value of $\sin 10° \cdot \sin 50° \cdot \sin 70°$.

   Answer: $\frac{1}{8}$.

   Solution. Transforming product of trigonometric functions into their sum we get $\sin 10° \sin 50° = \frac{1}{2}\cos 40° - \frac{1}{2}\cos 60° = \frac{1}{2}\cos 40° - \frac{1}{4}$. Therefore, the initial expression is equal to $\left(\frac{1}{2}\cos 40° - \frac{1}{4}\right)\sin 70° = \frac{1}{2}\cos 40° \sin 70° - \frac{1}{4}\sin 70° = \frac{1}{4}\sin 110° + \frac{1}{4}\sin 30° - \frac{1}{4}\sin 70°$. As $\sin 110° = \sin\left(180° - 70°\right) = \sin 70°$, we finally get $\frac{1}{4}\sin 30° = \frac{1}{8}$.

4. Find the coordinates of point M such that it lies on y-axis and tangent lines drawn from point M to parabola $y = 1+6x-4x^2$ are perpendicular to each other.

   Answer: $\left(0; \frac{53}{16}\right)$.

   Solution. Let us choose a point with ordinate a on y-axis. It is obvious that the lines we are considering are not parallel to coordinate axes. Then product of slopes of these lines equals (-1). Let us denote these slopes as k and $-\frac{1}{k}$ respectively. Consequently, the equations of the lines are $y = kx+a$ and $y = -\frac{1}{k}x+a$. Each of them has exactly one common point with the parabola, therefore systems

$$\begin{cases} y = 1+6x-4x^2, \\ y = kx+a \end{cases} \quad \text{and} \quad \begin{cases} y = 1+6x-4x^2, \\ y = -\frac{1}{k}x+a \end{cases}$$

   have one solution each. Equating the right sides of the equations in each of the systems, we get that equations

$$kx+a = 1+6x-4x^2 \quad \text{and} \quad -\frac{1}{k}x+a = 1+6x-4x^2$$

   must have exactly one solution each.

Let us consider the first equation separately. It is equivalent to $4x^2 + (k-6)x + (a-1) = 0$. For it to have one solution, its discriminant has to be 0, and so $(k-6)^2 - 16(a-1) = 0$. In the same way second equation yields $\left(\frac{1}{k}+6\right)^2 - 16(a-1) = 0$.

As k and a must satisfy both of the obtained equations, we can subtract the second of them from the first, thus getting $(k-6)^2 - \left(\frac{1}{k}+6\right)^2 = 0$; hence $k-6 = -\frac{1}{k}-6$ or $k-6 = \frac{1}{k}+6$. The former equation yields $k^2 = -1$, and so it has no solutions. The latter equation is equivalent to $k^2 - 12k - 1 = 0$, thus $k = 6 \pm \sqrt{37}$. Substituting these values of k into $(k-6)^2 = 16(a-1)$ we get $16(a-1) = 37$, $a = \frac{53}{16}$.

5. Point P belongs to side LM of parallelogram $KLMN$. It is known that $LP = MP = 3$, $\angle KPN = \arccos\frac{8}{9}$, $KL = 5$. Find the area of this parallelogram.

Answer: $2\sqrt{17}$.

Solution. Let $\angle KLM = \varphi$, then $\angle LMN = 180^\circ - \varphi$. Cosine theorem applied to triangles KLP and MNP yields $KP^2 = 25 + 9 - 30\cos\varphi$, $NP^2 = 25 + 9 + 30\cos\varphi$. Then we use cosine theorem for triangle KPN:

$$KN^2 = (34 + 30\cos\varphi) + (34 - 30\cos\varphi) - 2\sqrt{(34+30\cos\varphi)(34-30\cos\varphi)}\cdot\frac{8}{9}$$
$$\Leftrightarrow \sqrt{34^2 - 30^2\cos^2\varphi} = 18 \Leftrightarrow \cos^2\varphi = \frac{13\cdot 16}{15^2}.$$

Hence $\sin^2\varphi = 1 - \frac{13\cdot16}{15^2} = \frac{17}{15^2}$, $\sin\varphi = \frac{\sqrt{17}}{15}$, and so the area of the parallelogram is equal to $KL \cdot LM \cdot \sin\varphi = 2\sqrt{17}$.

6. Sketch the set of points whose coordinates $(x; y)$ satisfy the system of inequalities

$$\begin{cases} \log_{|x+2|-2|x+3|+4}(y-2) > \log_{|x+2|-2|x+3|+4}(1-x), \\ y < 13. \end{cases}$$

Find the area of this set.

Answer: The set is a union of three trapezoids $ABCD$, $APQR$ and $KLMR$, their borders do not belong to the set; the coordinates of the vertices are $A(-1;\, 4)$, $B(-1;\, 2)$, $C(0;\, 2)$, $D(0;\, 3)$, $P(-1;\, 13)$, $Q(-7;\, 13)$, $R(-7;\, 10)$, $K(-8;\, 11)$, $L(-8;\, 2)$, $M(-7;\, 2)$. The area of this set is 49.

Solution. Let us consider the two options available.

(a) The base of the logarithm is greater than 1. It happens if and only if

$$|x+2| - 2|x+3| + 4 > 1 \Leftrightarrow |2x+6| < |x+2| + 3 \Leftrightarrow \begin{cases} 2x+6 < |x+2|+3, \\ 2x+6 > -|x+2|-3 \end{cases} \Leftrightarrow$$

$$\begin{cases} |x+2| > 2x+3, \\ |x+2| > -2x-9 \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} x+2 > 2x+3, \\ x+2 < -2x-3, \end{array}\right. \\ \left[\begin{array}{l} x+2 > -2x-9, \\ x+2 < 2x+9 \end{array}\right. \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} x < -1, \\ x < -\frac{5}{3}, \end{array}\right. \\ \left[\begin{array}{l} x > -\frac{11}{3}, \\ x > -7 \end{array}\right. \end{cases} \Leftrightarrow \begin{cases} x < -1, \\ x > -7 \end{cases} \Leftrightarrow -7 < x < -1.$$

We get that $y - 2 > 1 - x$, i.e. $y > 3 - x$. If we also take into account that $y < 13$, we obtain a trapezoid $APQR$, the coordinates of its vertices being $A(-1;\, 4)$, $P(-1;\, 13)$, $Q(-7;\, 13)$, $R(-7;\, 10)$.

(b) The base of the logarithm is between 0 and 1. To determine the values of x for which it happens we solve the inequality $|x+2| - 2|x+3| + 4 > 0$ and then exclude an interval $[-7; -1]$ from its solution set.

$$|x+2|-2|x+3|+4>0 \Leftrightarrow |2x+6|<|x+2|+4 \Leftrightarrow \begin{cases} 2x+6<|x+2|+4, \\ 2x+6>-|x+2|-4 \end{cases} \Leftrightarrow$$

$$\begin{cases} |x+2|>2x+2, \\ |x+2|>-2x-10 \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} x+2>2x+2, \\ x+2<-2x-2, \end{array}\right. \\ \left[\begin{array}{l} x+2>-2x-10, \\ x+2<2x+10 \end{array}\right. \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} x<0, \\ x<-\dfrac{4}{3}, \end{array}\right. \\ \left[\begin{array}{l} x>-4, \\ x>-8 \end{array}\right. \end{cases} \Leftrightarrow \begin{cases} x<0, \\ x>-8 \end{cases} \Leftrightarrow -8<x<0.$$

Thus the base of the logarithm is between 0 and 1 for $x \in (-8;-7) \cup (-1;\,0)$. For these values of x we conclude that $y-2<1-x$, i.e. $y<3-x$. If we add that $y>2$ (that follows from the domain of the first inequality), we obtain two trapezoids $ABCD$ and $KLMR$, their vertices situated in points $A(-1;\,4)$, $B(-1;\,2)$, $C(0;\,2)$, $D(0;\,3)$, $R(-7;\,10)$, $K(-8;\,11)$, $L(-8;\,2)$, $M(-7;\,2)$.

The area of this set is equal to sum of areas of all the trapezoids. $S_{ABCD}=\frac{1+2}{2}\cdot 1=1{,}5$, $S_{APQR}=\frac{9+3}{2}\cdot 6=36$, $S_{KLMR}=\frac{9+8}{2}\cdot 1=8{,}5$; hence $S_{\text{total}}=46$.

7. Isosceles trapezoids $APRS$ and $PQRS$, their largest bases being PR and PS respectively ($PR=PS$), are inscribed into circle Ω. Diagonals of trapezoid $APRS$ intersect at point O, and angle AOP equals 60°. Find the radius of Ω given that the area of triangle PQR is equal to $9+9\sqrt{3}$.

   Answer: 6.

   Solution. Angle AOP is the angle between the chords that intersect; therefore, it is equal to one half of the sum of arcs AP and RS. As these arcs are equal to each other (they are arcs between parallel chords PR and AS), we conclude that each of them is equal to 60°. Chords QR and PS are also parallel to each other (as they are bases of a trapezoid), and so arc PQ is also equal to 60°. Chords PR and PS are equal to each other; and so are the corresponding arcs, that is, arc PQR equals arc PAS. From here follows that $\smile QR=\smile AS$. As the whole circle is 360°, we get that $\smile QR=\smile AS=90^\circ$.

   Angle QPR is inscribed into circle Ω, and so it is equal to $\frac{1}{2}\smile QR=\frac{1}{2}90^\circ=45^\circ$. In the same way $\angle PRQ=\frac{1}{2}\smile PQ=30^\circ$. Then $\angle PQR=180^\circ-\angle QPR-\angle PRQ=105^\circ$. Let the diameter of the circle be equal to d. Then sine theorem used for triangle PQR yields $PQ=d\sin 30^\circ=\frac{d}{2}$, $QR=d\sin 45^\circ=\frac{d}{\sqrt{2}}$. For expressing the area of triangle PQR we also need that $\sin 105^\circ=\sin(45^\circ+60^\circ)=\sin 45^\circ\cos 60^\circ+\cos 45^\circ\sin 60^\circ=\frac{1+\sqrt{3}}{2\sqrt{2}}$.

   Hence area of triangle PQR is equal to $\frac{1}{2}PQ\cdot QR\cdot\sin 105^\circ=\frac{d^2\left(\sqrt{3}+1\right)}{16}$. Equating it to $9\left(\sqrt{3}+1\right)$ yields $d^2=144$, so $d=12$ and the radius is equal to 6.