International Scientific Physics Olympiad

T3. Frequency measurements with ring resonators

A1 What are the coefficients C_1 and C_2?

They are equal to 0 because the field cannot tend to infinity at a large distance from the fiber.

$$C_1 = 0$$
$$C_2 = 0$$

A2 Express k_{in} and k_{out} in terms of $\varepsilon, \omega, \beta$ and speed of light in vacuum $c = 1/\sqrt{\varepsilon_0 \mu_0}$, using the wave equation.

Substituting solution type into the wave equation:

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + (\omega^2 \varepsilon / c^2 - \beta^2) A = 0$$

Considering that $\frac{\partial^2 A}{\partial x^2} \ll \frac{\partial^2 A}{\partial y^2}$, we get:

$$k_{in} = \sqrt{\frac{\omega^2 \varepsilon(\omega)}{c^2} - \beta^2}$$
$$k_{out} = \sqrt{\beta^2 - \frac{\omega^2}{c^2}}$$

A3 Prove that the tangential components of the electric field strength are equal on both sides of the interface (as in electrostatics).

We can write Faraday's law of electromagnetic induction for a rectangular circuit with vertices at points $(a/2 - \delta a, y, z), (a/2 - \delta a, y, z + \Delta z), (a/2 + \delta a, y, z), (a/2 + \delta a, y, z)$.At $\delta a \to 0$, the derivative of the magnetic field flux tends to 0, and the magnitude of the circulation of the electric field strength remains constant. From this we can conclude that it is also equal to 0. At $\delta a \to 0$ the circulation can be expressed as $\Gamma = (E_{\tau 1} - E_{\tau 2}) \Delta z = 0$, hence $E_{\tau 1} = E_{\tau 2}$.

A4 Write the boundary conditions for the tangential component of the electric field. Express B in terms of a, k_{in}, k_{out}.

$$B = \exp(k_{out} a/2) \cos(k_{in} a/2)$$

A5 Prove that the tangential components of the magnetic field strength are equal on both sides of the interface (as in magnetostatics).

The proof is similar to A3, but instead of the law of electromagnetic induction, the circulation theorem for the magnetic field strength is used.

A6 Prove that

$$H_z = E_0 \operatorname{Re} \left(\frac{i}{\mu_0 \omega} \frac{\partial A}{\partial x} \exp(i(\omega t - \beta z)) \right).$$

Using the differential form of Faraday law we get:

$$-\mu_0 \frac{\partial H_z}{\partial t} = \frac{\partial E_y}{\partial x}$$

The derivative of H_z with respect to time can be expressed as:

$$\frac{\partial H_z}{\partial t} = i \omega H_z$$

$$H_z = E_0 \operatorname{Re} \left(\frac{i}{\mu_0 \omega} \frac{\partial A}{\partial x} \exp(i \omega t - i \beta z) \right)$$

A7 Write the boundary conditions for the tangential component of the magnetic field $\vec{H}$. The answer can be expressed in terms of k_{in}, k_{out}, a.

To find H_z, differentiate A by x:

$$\frac{\partial A}{\partial x} = -k_{in}\sin(k_{in}x), |x| < a/2$$

$$\frac{\partial A}{\partial x} = -k_{out}\exp(k_{out}(-a/2+x))\cos(k_{in}a/2), |x| > a/2$$

Equating H_z on both sides of the interface:

$$\tan(k_{in}a/2) = \frac{k_{out}}{k_{in}}$$

A8 Substitute the values k_{in}, k_{out}, obtained in A2 into the equation from A7. Obtain an equation from which βcan be determined (this equation is solved numerically only). The equation can include $\beta, \omega, \varepsilon, c, a$.

$$\tan\left(\frac{a}{2}\sqrt{\frac{\omega^2\varepsilon(\omega)}{c^2} - \beta^2}\right) = \sqrt{\frac{\beta^2c^2 - \omega^2}{\omega^2\varepsilon(\omega) - \beta^2c^2}}$$

B1 Assuming that there is no energy loss in the divider, find the relationship between r_s and t_s.

$$r_s^2 + t_s^2 = 1$$

B2 Express the field amplitude at the input to Q_1 of the splitter $B_{in}(t)$ in terms of κ and the field amplitude at the output of Q_2 at time $(t-\tau(\omega))$ — $B_{out}(t-\tau(\omega))$.

$$B_{in}(t) = \kappa B_{out}(t-\tau(\omega))$$

B3 Using the stationarity conditions, express $B_{in}(t)$ in terms of $B_{out}(t), \kappa, \omega, \tau$.

Let the φ be:

$$\varphi = -\omega\tau$$

Then

$$B_{out}(t) = it_sA_{in}(t) - r_sB_{in}(t)$$

$$B_{in}(t) = \kappa e^{i\varphi}B_{out}(t)$$

$$B_{in}(t) = \kappa e^{-i\omega\tau}B_{out}(t)$$

B4 Express $B_{in}(t)$ in terms of $A_0, r_s, t_s, \kappa, \omega, \tau$ and t, using the result of the previous task.

$$B_{in}(t) = \kappa e^{i\varphi}(it_sA_{in}(t) - r_sB_{in}(t))$$

$$B_{in}(t) = \frac{it_s\kappa e^{i\varphi}A_{in}(t)}{1 + r_s\kappa e^{i\varphi}}$$

$$B_{in}(t) = \frac{it_s\kappa e^{-i\omega\tau}A_0e^{i\omega t}}{1 + r_s\kappa e^{-i\omega\tau}}$$

B5 What is the power N_2, leaving the channel P_2? Express the answer in terms of $\omega\tau(\omega), \kappa, r_s$ and the power N_1 in channel P_1.

$$A_{out}(t) = -r_s A_{in}(t) + it_s B_{in}(t) = -A_{in}(t)\left(r_s + \frac{t_s^2 \kappa e^{i\varphi}}{1 + r_s \kappa e^{i\varphi}}\right)$$

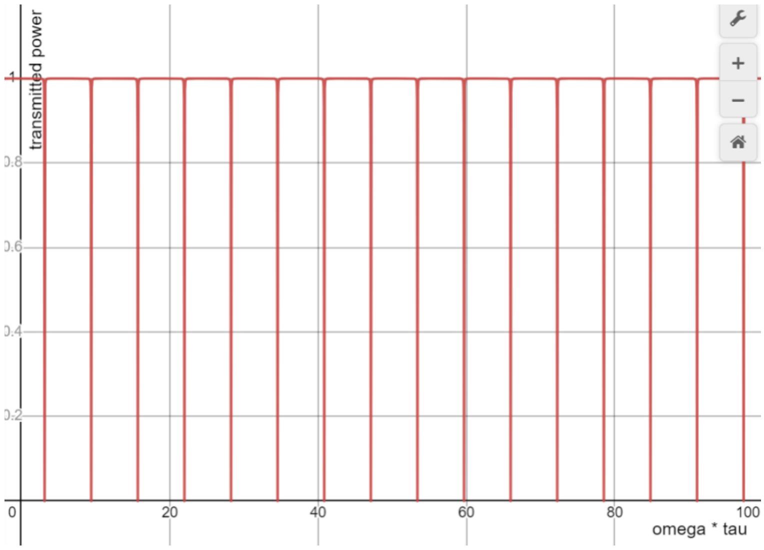
Let us reduce the right-hand expression to a common denominator and divide the square of the numerator modulus by the square of the denominator modulus:

$$\eta = \frac{N_2}{N_1} = \frac{\kappa^2 \sin^2(\omega\tau) + (r_s + \kappa\cos(\omega\tau))^2}{\kappa^2 r_s^2 \sin^2(\omega\tau) + (1 + r_s\kappa\cos(\omega\tau))^2}$$

B6 Sketch a qualitative plot of $N_2/N_1(\omega\tau)$ for fiber resonator with the following parameters:

- $\kappa = 1 - 5 \cdot 10^{-3}$;
- $t_s = 0.1$.

At what values of $\omega\tau$ is the ratio N_2/N_1 minimal?



$$\omega_{res}\tau = \pi(1 + 2n)$$

B7 Find the sharpness Q of the absorption peak with number $n = 100$.
Sharpness is the ratio of the peak frequency to the width of the region of frequencies for which the transmission dip is not less than half of the maximum dip of a particular peak.

Let us find the depth of the absorption peak by expanding the expression for the power ratio into a Taylor series:

$$\eta(\omega_{res}) \approx \frac{(1 - \kappa - t^2/2)^2}{(1 - \kappa + t^2/2)^2} \approx 0$$

$$\Delta\eta_{max} = 1 - \eta(\omega_{res}) \approx 1$$

Or by substituting the value φ from the previous paragraph, without putting it in a row:

$$\eta(\omega_{res}) \approx 1.6 \cdot 10^{-6} \approx 0$$

$$\Delta\eta_{max} \approx 1$$

Sharpness is the ratio of ω to the width of the region in which $\Delta\eta > \Delta\eta_{max}/2$. Let's estimate the width of this area:

$$\varphi = \delta + \pi(1 + 2n), \delta \ll \pi$$

Let us expand the Maclaurin series in terms of δ:

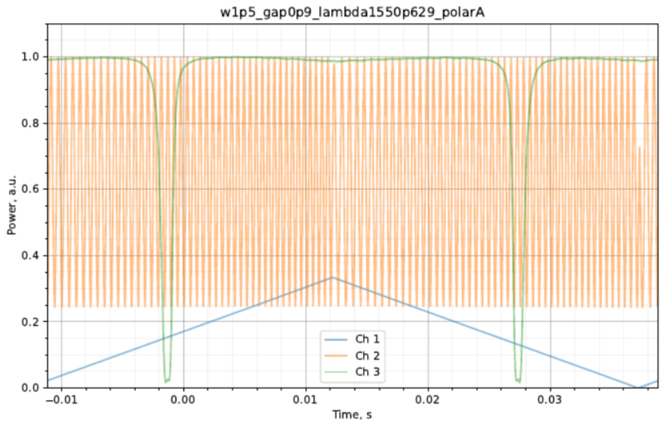
$$\eta \approx \frac{\delta^2 + (1-\kappa - t^2/2 + \delta^2/2)^2}{\delta^2 + (1-\kappa + t^2/2 + \delta^2/2)^2} \approx \frac{\delta^2}{\delta^2 + (1-\kappa + t^2/2)^2} = \frac{1}{2}$$

From here we get:

$$\delta \approx 1 - \kappa + t^2/2 \approx 0.01$$

$$Q = \frac{\omega_{res}}{\Delta\omega} = \frac{\pi(1+2n)}{2\delta} \approx \frac{\pi n}{1-\kappa+t^2/2} = \pi \cdot 10^4 = 3.14 \cdot 10^4$$

C1 Sketch a qualitative plot of the oscilloscope readings if it is known for sure that the frequency of the tunable laser reaches exactly one of the frequencies given in B6 (these are the frequencies where the FR transmittance is minimal).

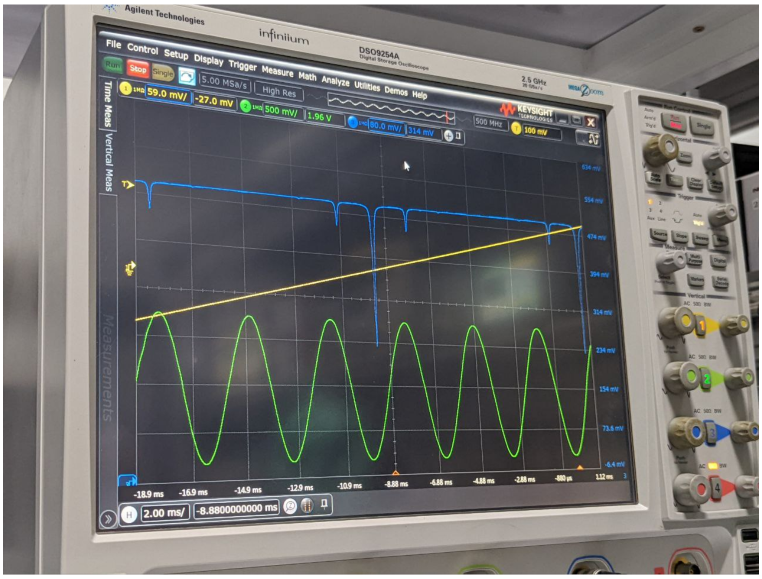


C2 Draw what the oscilloscope will show when $\Omega \approx 220 MHz$. Note that $\alpha \ll \Omega\omega_0$.

The field amplitude at the exit from the EOM will be equal to

$$E_{EOM}(t) = f(t)\cos(\omega(t)t) = \beta\cos(\omega t) + \frac{1-\beta}{2}\Big(\cos(\Omega+\omega)t + \cos(-\Omega+\omega)t\Big)$$

where $\omega = \omega(t) = \omega_0 + \alpha t$. In this case, three waves fall on the FR, the frequencies of which are shifted relative to each other by exactly Ω. Therefore, the number of absorption peaks will now triple: they can be observed at frequencies $\omega_{res}, \omega_{res}+\Omega, \omega_{res}-\Omega$, where ω_{res} is frequency corresponding to the minimum transmission (defined in B6).



C3 Estimate with what relative accuracy can the period ω_{MZI} be measured on this setup if the maximum signal frequency of the high-frequency oscillator is $\Omega_{max} = 1250MHz$?

We can measure one period by changing Ω in a way, so that the absorption maxima were separated by ω_{max} from each other. Then the absolute error would be of order $\omega_0/Q = 24MHz$, and relative $\omega_0/(Q\Omega_{max}) = 0.02$

D1 Express the group velocity v_g in terms of β_1.

$$v_g = 1/\beta_1$$

D2 Let us find the form of the soliton. The solution of NLSE can be found in the form:

$$F(z, s) = \frac{F_0 \exp(i\sigma z)}{\cosh(\theta s)}.$$

Experess F_0 и σ in terms of θ, β_1, β_2 и γ.

Let's calculate the derivatives and substitute them into the equation (reducing by $iF_0 \exp(i\sigma z)$):

$$\frac{\sigma}{\cosh\theta s} + \frac{\beta_2\theta^2}{2\beta_1^2}\Big(-\frac{2}{\cosh^3\theta s} + \frac{1}{\cosh\theta s}\Big) = \frac{\gamma F_0^2}{\cosh^3\theta s}$$

From here we get:

$$F_0^2 = -\frac{\beta_2\theta^2}{\gamma\beta_1^2}$$

$$\sigma = -\frac{\beta_2\theta^2}{2\beta_1^2}$$

D3 Express D_1 and D_2 in terms of β_1, β_2 and loop length L. *Hint: the BP natural frequency criterion:* $B_{in}(t)$ *and* $B_{in}(t - \tau(\omega_\mu))$ *have the same phase.*

Using the result of point B3, we get:

$$\beta(\omega(\mu))L + \pi = 2\pi(\mu + \mu_0)$$

Now let's substitute the expression for $\beta(\omega)$, and into it $\omega(\mu)$. We get:

$$\Big(\beta_0 + \beta_1(D_1\mu + \frac{D_2\mu^2}{2}) + \frac{\beta_2 D_1^2\mu^2}{2}\Big)L \approx 2\pi(\mu + \mu_0 + 1/2)$$

Equating the coefficients of μ and μ^2 on the right and left sides:

$$D_1 = \frac{2\pi}{\beta_1 L}$$

$$D_2 = -\frac{\beta_2 D_1^2}{\beta_1} = -\frac{4\pi^2\beta_2}{\beta_1^3 L^2}$$

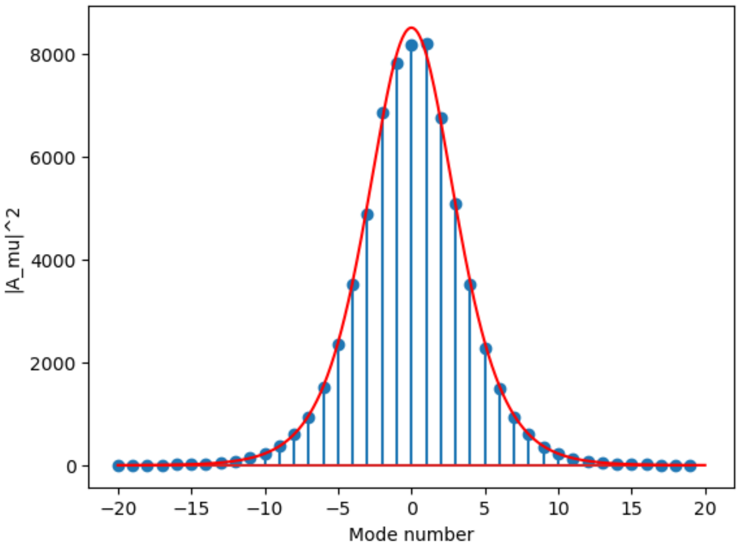
D4 Let the resonator be made of a material with $\chi > 0$. At what D_2 can solitons exist in it?

$F_0^2 > 0$ so $\beta_2 < 0$ and $D_2 > 0$

$$D_2 > 0$$

D5 Let a soliton with carrier frequency ω_0. circulates in the FR described in D3. The external laser does not work. Plot the emission spectrum of the resonator (the dependence of specific power on frequency) **qualitatively** in the frequency range $(\omega_0 - 20D_1, \omega_0 + 20D_1)$. Consider that $\omega_0/Q(\omega_0) \ll D_1$. (Remember that Q -is sharpness defined in B7)

Spectrum consists of narrow bands corresponding to the natural frequencies of the FR as shown on the figure. The envelope can be fitted by $F_1/\cosh^2(F_2(\omega - \omega_0))$ (here F_1, F_2 are some constants depending on F_0, σ, θ), but analytical expression for envelope form is not required.



D6 Estimate the absolute error of the angular frequency measurement ω using the spectrum from item D5.

We get a ruler with a pitch of D_1, and the absolute error is about a half of this pitch – $D_1/2$.

$$\Delta\omega \approx \frac{D_1}{2}$$

D7 Express the round-trip time τ_s through D_1.

$$\tau_s = \frac{2\pi}{D_1}$$