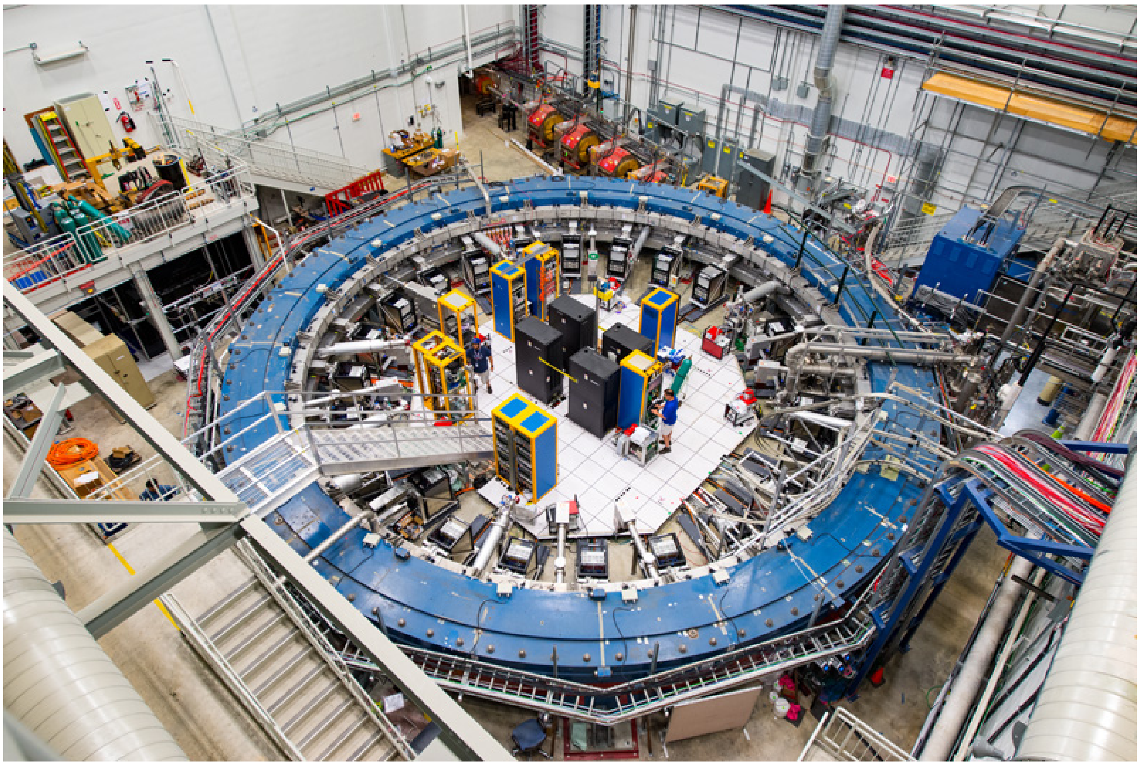
T2. Muon anomalous magnetic moment

Measurement of electron and muon magnetic moments is one of the most accurate in modern physics. This measurement provides a very precise test for elementary particles model. In the experiment positively charged muons (also called antimuons) are used. Antimuon mass is approximately 200 times larger than mass of the electron and antimuon charge is equal to the absolute value of electron charge. Throughout the problem we will refer to the positively charged muons as just muons for brevity.

Muon (and electron) has angular momentum $\vec{s}$, which is not connected with its motion in space. This angular momentum is called spin of the muon. Spin motion can be described by the torque equation, the rate of change of spin moment is equal to the applied torque. Moreover, muon has magnetic moment

$$\vec{\mu} = g_\mu \frac{q}{2m_\mu}\vec{s}.$$

Here m_μ and q are mass and charge of the muon, $g_\mu = 2(1 + a_\mu)$ is a dimensionless constant. The parameter a_μ describes the deviation of magnetic moment from «normal» value (which corresponds to $g_\mu = 2$). The problem deals with an experiment in which the anomalous contribution to magnetic moment a_μ is determined. Muons move in a circle in homogeneous magnetic field and their magnetic moment precession frequency is measured .



Storage ring for muon $g - 2$ measurement experiment, Fermilab

In particle physics energies are usually measured in electronvolts (eV). Instead of particle mass m the rest energy mc^2 is used, its value is also measured in electronvolts. You may use the following numerical values:

- 1 eV $= 1.602 \cdot 10^{-19}$ J
- Speed of light in vacuum $c = 2.998 \cdot 10^8$ m/s
- Muon mass $m_\mu\, c^2 = 105.7$ MeV
- Antimuon charge $q = 1.602 \cdot 10^{-19}$ C
- Muon is unstable, its lifetime in the rest frame is $\tau = 2.197 \cdot 10^{-6}$ s

Rotation can be described by an angular velocity vector $\vec{\omega}$, which is directed along the rotation axis, the direction of $\vec{\omega}$ is determined by the right-hand rule. If a vector $\vec{A}$ rotates in laboratory frame of reference with angular velocity $\vec{\omega}$, its time derivative is

$$\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A}.$$

Let the reference frame B (coordinates and time are x', y', z', t') move with respect to laboratory reference frame A (coordinates and time ar x, y, z, t) with velocity V along x axis. The coordinates in these frames are related by the Lorentz transformation

$$x = \gamma\left(x' + Vt'\right), \quad t = \gamma\left(t' + \frac{V}{c^2}x'\right), \quad y = y', \quad z = z', \quad \gamma = \frac{1}{\sqrt{1 - V^2/c^2}}.$$

In all your answers you can use the γ defined above to simplify formulas.

If particle's velocity in the moving reference frame B is $\vec{v}'$, with projections on coordinate axes are v'_x, v'_y, v'_z, its velocity components in the laboratory reference frame are

$$v_x = \frac{v'_x + V}{1 + \frac{v'_x V}{c^2}}, \quad v_y = \frac{v'_y\sqrt{1 - V^2/c^2}}{1 + \frac{v'_x V}{c^2}}, \quad v_z = \frac{v'_z\sqrt{1 - V^2/c^2}}{1 + \frac{v'_x V}{c^2}}.$$

Energy and momentum of relativistic particle with mass m moving at speed $\vec{v}$ are

$$E = \gamma mc^2, \quad \vec{p} = \gamma m\vec{v}, \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

If particle moves under the force $\vec{F}$, the equation of motion is

$$\frac{d\vec{p}}{dt} = \vec{F}.$$

In questions that requier a numerical answer you need to calculate three significant figures.

Part A. Dynamics in magnetic field (4.5 points)

During anomalous magnetic moment measurement experiment a beam of muons moves in magnetic field $B = 1.450$ T, muons' momenta are perpendicular to the magnetic field and equal to $pc = 3.094$ GeV. The magnetic field is directed along the z axis.

A1	For a muon in the beam find the difference between speed of light and muons' speed $\Delta v = c - v$ (provide formula and compute numerical value).	**1.0**

A2	For a muon in the beam find the kinetic energy E_k, that is the difference between the full energy and the rest energy (provide formula and compute numerical value in Gev).	**0.5**

A3	Find radius R of muon's orbit and period T of its motion. (Provide formulas and compute numerical values.)	**1.0**

A4	Due to instability a muon can orbit the circle only finite time. Find number of revolutions N the muon makes before its decay.	**0.8**

Now we consider the case when muon is at rest in the same magnetic field B as in previous parts.

A5	Obtain a formula for the spin time derivative, $d\vec{s}/dt$, express the answer in terms of $\vec{B}$, $\vec{s}$, g_μ and fundamental constants.	**0.7**

A6	Find the z component of angular velocity ω_z of muon's spin. Express the answer in terms of B, g_μ and fundamental constants.	**0.5**

Part B. Thomas precession (3.5 points)

The muon spin is not retated to its spatial motion and should be described in the frame of reference in which the muon is at rest (such a frame is called comoving frame). If the muon is moving with acceleration, at each instant of time we must choose a different comoving frame of reference. Consider two successive Lorentz transformations. The first transformation is from the laboratory frame A (coordinates x, y, t) to the moving frame B (coordinates x', y', t'). The second is from the reference frame B to the third reference frame C (coordinates x'', y'', t''). It turns out that combination of this transformation is not a Lorentz transformation with some resulting velocity. There is an additional rotation which leads to the spin precession. This effect is called Thomas precession. Thomas precession is purely kinematics phenomenon, so to describe it we do not have to consider magnetic moment dynamics.

Let the reference frame B move with velocity v_x along x axis with respect to A. In turn reference frame C moves with velocity v'_y along y axis with respect to B. You may assume that $v'_y \ll v_x$ and work up to the first order in v'_y. To simplify your expressions you may use notation $\gamma = 1/\sqrt{1 - v_x^2/c^2}$.

B1	Find the x and y components of velocity $\vec{v}_C$ of reference frame C with respect to the reference frame A. Express the answer in terms of v_x, v'_y.	**0.5**

B2	Find the x'' and y'' components of velocity $\vec{v}''_A$ of the reference frame A with respect to the reference frame C. Take into account that v''_y is small.	**0.7**

B3	It turns out that $\vec{v}''_A \neq -\vec{v}_C$. It can be attributed to the fact that axes of the reference frame C are rotated with respect to the axes of frame A. Therefore all vectors in frame C are also rotated with respect to the frame A. Calculate the angle of rotation $\Delta\theta$ defined as an angle between vectors $-\vec{v}''_A$ and $\vec{v}_C$. The angle is positive if the rotation from first vector to the second vector viewed from the z axis is counterclockwise. (Axes x, y, z form right-handed coordinate system.) Express your answer in terms of v_x, v'_y, $\gamma = 1/\sqrt{1 - v_x^2/c^2}$, c.	**0.6**

B4	Let a particle move with velocity $\vec{v}$ along x axis with respect to the frame A. Particle's acceleration is directed along the y axis, its projection is a_y. Frame B is the comoving frame for the particle at the considered moment (i.e. the velocity of the particle in it is zero) and moves with the speed v along x axis. After time interval dt (measured in laboratory reference frame) particle velocity is $\vec{v} + d\vec{v}$, corresponding comoving reference frame is C. From the previous question it follows that the axes of comoving frame rotate. Using results of the previous quaestion, find z component ω_T (angular velocity of Thomas precession) of the comoving frame angular velocity (in this case rotation is around z axis). Express the answer in terms of v_x, a_y, γ.	**1.0**

B5	Let the muon move in homogeneous magnetic field B directed along z axis. Muon velocity is perpendicular to the magnetic field. Calculate the z component of Thomes precession angular velocity. Express the answer in terms of B, q, m_μ, γ.	**0.7**

Part C. Magnetic moment precession (2 points)

Let us consider muon moving in homogeneous magnetic field. Muon spin precession is determined by two factors: direct interaction with magnetic field and Thomas precession due to accelerated motion. The angular velocity of precession due to magnetic field can be calculated by the formula, obtained in the question A6 in the case of muon at rest. Далее везде будем считать, что спин мюона параллелен плоскости движения мюона.

C1	Find the expression for z component of muon spin precession ω_s angular velocity with respect to laboratory reference frame. Express the answer in terms of B, g_μ, q, m_μ, γ.	**0.7**

C2	In practice it is more convenient to consider spin rotation with respect to the direction of muon momentum. Find the angular velocity ω_a of muon spin rotation with respect to the direction of its momentum (that is also equal to the time derivative of the angle between momentum and spin). Express the answer in terms of B, a_μ, q, m_μ, γ.	**0.5**

C3	Experimentally measured frequency of muon spin precession with respect to the momentum direction is $f_a = 229081$ Hz. Find the expression for muon anomalous magnetic moment a_μ. Express the answer in terms of f_a, B, q, m_μ, γ. Compute the numerical value of a_μ.	**0.8**